The consistent reduction of the differential calculus on the quantum group $GL_q(2,C)$ to the differential calculi on its subgroups and σ-models on the quantum group manifolds $SL_q(2,R)$, $SL_q(2,R)/U_h(1)$, $Cq(2|0)$ and infinitesimal transformations

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Abstract

Explicit construction of the second order left differential calculi on the quantum group and its subgroups are obtained with the property of the natural reduction: the differential calculus on the quantum group $GL_q(2,C)$ has to contain the 3-dimensional differential calculi on the quantum subgroup $SL_q(2,C)$, the differential calculi on the Borel subgroups $B_L^{(2)}(C)$, $B_U^{(2)}(C)$ of the lower and of the upper triangular matrices, on the quantum subgroups $U_q(2)$, $SU_q(2)$, $Sp_q(2,C)$, $Sp_q(2)$, $T_q(2,C)$, $B_L(C)$, $B_U(C)$, $U_q(1)$, $Z_-^{(2)}(C)$, $Z_+^{(2)}(C)$ and on the their real forms. The classical limit ($q \to 1$) of the left differential calculus is the nondeformed differential calculus. The differential calculi on the Borel subgroups $B_L(C)$, $B_U(C)$ of the $SL_q(2,C)$ coincide with two solutions of Wess-Zumino differential calculus on the quantum plane $C_q(2|0)$.

The spontaneous breaking symmetry in the WZNW model with $SL_q(2,R)$ quantum group symmetry over two-dimensional nondeformed Minkovski space and in the σ-models with $SL_q(2,R)/U_p(1)$, $C_q(2|0)$ quantum group symmetry is considered. The Lagrangian formalism over the quantum group manifolds is discussed. The variational calculus on the $SL_q(2,R)$ group manifold is obtained. The classical solution of $C_q(2|0)$ σ-model is obtained.

I. Introduction

The left-(right-) invariant differential calculus on the $GL_q(2,C)$, matched with differential calculi on the $SL_q(2,C)$ and on the $C_q(2|0)$, was constructed in [1,2], the differential calculus on the $GL_q(2,C)$, matched with the differential calculi on its subgroups, was constructed in [3]. In this paper we shortly remember the basic states of the left-invariant consistent differential calculus and on this base we consider the two-dimensional WZNW model on the $SLq(2,R)$ group over nondeformed $M^{(1,1)}$ space. We obtained the expression for the background gravity and antisymmetric fields in the σ-model approach to a relativistic string. Also, we considered the spontaneous breaking symmetry in this model and in the σ-models on the quantum group manifolds $SL_q(2,R)/U_h(1)$ and on the $C_q(2|0)$. We considered the infinitesimal transformations on the $SL_q(2,R)$ and obtained the commutation relations of the variational calculus on this group.

II. Differential calculus on the $GL_q(2,C)$ group.

The matrix quantum group [1] $G = GL_q(2,C)$ is defined by the q-commutation relations (C.R.) of its group parameters. Let $g \in GL_q(2,C)$

$$g = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad R^{ij}_{kl} g^k_m g^l_n = g^j_l g^i_k R^{kl}_{mn} \tag{1}$$

The Rgg-relations in components have the standard form

$$\begin{array}{llll} ab = qba, & bd = qdb, & bc = cb, & \lambda = q - \frac{1}{q} \\ ac = qca, & cd = qdc, & ad = da + \lambda bc & \end{array} \tag{2}$$

The quantum determinant D_q is the central element and commutes with the group elements

$$det_q g = D_q = ad - qbc = da - \frac{1}{q}cb, \quad D_q g_i^k = g_i^k D_q \tag{3}$$

The left-side exterior derivative of any function f of group parameters is

$$df = \left(f \frac{\overleftarrow{\partial}}{\partial g_i^k} \right) dg_i^k \tag{4}$$

The differential calculus on a quantum group is defined by C.R. between the group parameters and theirs differentials, or between the group parameters and Maurer-Cartan 1-forms

$$\omega = g^{-1}dg = \begin{pmatrix} \omega_1^1 & \omega_2^1 \\ \omega_1^2 & \omega_2^2 \end{pmatrix} = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} \tag{5}$$

The basic states of the left-invariant differential calculus are following:

1. Following [5], we look for C.R. between 1-forms and group parameters in the form

$$\omega_\beta^\alpha T_k^i = T_m^i \left(F_k^m\right)_{\delta\beta}^{\alpha\gamma} \omega_\gamma^\delta, \quad F_k^m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

and A, B, C, D are 4×4 unknown matrices with complex entries.

2. The requirement

$$[\omega_j^i, \left(R_{\beta m}^{\alpha n} T_\delta^\beta T_k^m - T_m^n T_\beta^\alpha R_{\delta k}^{\beta m} \right)] = 0 \tag{7}$$

leads to the equations for the matrices A, B, C, D

$$R_{\beta m}^{\alpha n} F_\delta^\beta F_k^m = F_m^n F_\beta^\alpha R_{\delta k}^{\beta m} \tag{8}$$

A, B, C, D are the representation of a, b, c, d.

3. For the quantum determinant D_q we define the quantum trace by the definition

$$dD_q = D_q \mathrm{Tr}_q \omega \tag{9}$$

We obtained

$$\begin{aligned} dD_q = D_q \left(\omega^1 + A_l^4 \omega^l - \tfrac{1}{q} B_l^3 \omega^l \right) \quad &= D_q \left(\omega^4 + D_l^1 \omega^l - q C_l^2 \omega^l \right) \\ = D_q \left(\omega^1 + A_l^4 \omega^l - \tfrac{1}{q} C_l^2 \omega^l \right) \quad &= D_q \left(\omega^4 + D_l^1 \omega^l - q C_l^2 \omega^l \right) \end{aligned} \tag{10}$$

and additional conditions

$$\omega^2 + B_l^1 \omega^l - q A_l^2 \omega^l = 0, \quad \omega^3 + C_l^4 \omega^l - \frac{1}{q} D_l^3 \omega^l = 0 \tag{11}$$

4. The C.R. between 1-forms and the quantum determinant are

$$\omega_k^i D_q = D_q \left(AD - qBC\right)_m^i \omega_k^m \tag{12}$$

5. Differentiating the commutation relations between 1-forms and group parameters and using the Maurer–Cartan equation $d\omega = -\omega \wedge \omega$ we have the q-deformation of the algebra of the ω-forms:

$$\left(F_m^f\right)_{\tau\gamma}^{\alpha\varphi} \left(F_k^m\right)_{\sigma\beta}^{\gamma\rho} \omega_\varphi^\tau \omega_\rho^\sigma + \left(F_\sigma^f\right)_{\tau\beta}^{\alpha\varphi} \omega_\varphi^\tau \omega_k^\sigma + \left(F_k^\varphi\right)_{\sigma\beta}^{\alpha\rho} \omega_\varphi^f \omega_\rho^\sigma - \left(F_k^f\right)_{\tau\beta}^{\alpha\rho} \omega_\sigma^\tau \omega_\rho^\sigma = 0 \tag{13}$$

All of this conditions are common for any differential calculus.

6. The last condition is the basic condition, which differs the matched differential calculus from the bicovariant differential calculus [6]. Instead of bicovariance condition we require

$$\omega^k D_q = D_q \omega^k \tag{14}$$

which leads to the equation $AD - qBC = 1$. To solve the system of equations for the matrix F satisfying to this conditions we considered the following representation of algebra (8) [1]

$$B = C = 0 \qquad AD = 1. \tag{15}$$

We obtained the following expressions for the quantum trace

$$dD_q = D_q \left[D^1_1 \omega^1 + (1 + D^1_4)\omega^4 \right] = D_q \left[(1 + A^4_1)\omega^1 + A^4_4 \omega^4 \right], \tag{16}$$

where we used the composite indexes 1,2,3,4. Finally, we found the matrices A, D in terms of the independent variables $\beta = D^1_1,\ \alpha = A^4_4$

$$A = \begin{pmatrix} 1 - \alpha + \frac{\alpha}{\beta} & 0 & 0 & \frac{(1-\alpha)\alpha}{\beta} \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ \beta - 1 & 0 & 0 & \alpha \end{pmatrix}, \quad D = \begin{pmatrix} \beta & 0 & 0 & \alpha - 1 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ \frac{(1-\beta)\beta}{\alpha} & 0 & 0 & 1 - \beta + \frac{\beta}{\alpha} \end{pmatrix} \tag{17}$$

We found, that the algebra of the 1-forms can be decomposed on the algebra $SL_q(2, C)$ and the $U(1)$ subalgebra only for unique value of the parameters

$$\alpha = \frac{2}{1 + q^2}, \qquad \beta = \frac{2q^2}{1 + q^2} \tag{18}$$

In this case the commutation relations between the parameters and the 1-forms are diagonalized after the choice of the new basis of 1-forms

$$\bar{\omega}^1 = \frac{2}{q + 1/q}(q\omega^1 + \frac{1}{q}\omega^4) = \mathrm{Tr}_q \omega, \quad \bar{\omega}^4 = \frac{1}{1 + q^2}(\omega^1 - \omega^4) \tag{19}$$

$$\begin{array}{ll} \bar{\omega}^1 a = a\bar{\omega}^1 & \bar{\omega}^1 d = d\bar{\omega}^1 \\ \bar{\omega}^4 a = q^{-2} a\bar{\omega}^4 & \bar{\omega}^4 d = q^2 d\bar{\omega}^4, \\ \omega^2 a = q^{-1} a\omega^2 & \omega^2 d = qd\omega^2, \\ \omega^3 a = q^{-1} a\omega^3 & \omega^3 d = qd\omega^3 \end{array} \tag{20}$$

The other relations are found by making the interchanges $a \leftrightarrow c, d \leftrightarrow b$. The commutation relations between 1-forms can be written as

$$\begin{array}{ll} \bar{\omega}^1 \omega^2 + \omega^2 \bar{\omega}^1 = 0, & \bar{\omega}^1 \omega^3 + \omega^3 \bar{\omega}^1 = 0 \\ q^2 \omega^2 \bar{\omega}^4 + q^{-2} \bar{\omega}^4 \omega^2 = 0, & q^2 \bar{\omega}^4 \omega^3 + q^{-2} \omega^3 \bar{\omega}^4 = 0 \\ \bar{\omega}^1 \bar{\omega}^4 + \bar{\omega}^4 \bar{\omega}^1 = 0, & \omega^2 \omega^3 + q^{-2} \omega^3 \omega^2 = 0 \end{array} \tag{21}$$

$$(\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 = (\omega^4)^2 = 0 \tag{22}$$

[1] Note,that the solutions for the matrix F with $B, C \neq 0$, considered by author, lead to the differential calculi which have not classical limit ($q \to 1$).

The algebra of the vector fields $\overleftarrow{\nabla}_k$ has the form

$$\begin{array}{cc} \overleftarrow{\nabla}_3\overleftarrow{\nabla}_2 - q^2\overleftarrow{\nabla}_2\overleftarrow{\nabla}_3 = \overleftarrow{\nabla}_1 & q^2\overleftarrow{\nabla}_2\overleftarrow{\nabla}_1 - q^{-2}\overleftarrow{\nabla}_1\overleftarrow{\nabla}_2 = (1+q^2)\overleftarrow{\nabla}_2 \\ q^2\overleftarrow{\nabla}_1\overleftarrow{\nabla}_3 - q^{-2}\overleftarrow{\nabla}_3\overleftarrow{\nabla}_1 = (1+q^2)\overleftarrow{\nabla}_3 & [\overleftarrow{\nabla}_4, \overleftarrow{\nabla}_k] = 0 \end{array} \tag{23}$$

After the mapping

$$\overleftarrow{\nabla}_1, \overleftarrow{\nabla}_2, \overleftarrow{\nabla}_3, \overleftarrow{\nabla}_4 \to H, T_2, T_3, N \tag{24}$$

$$\begin{array}{cc} \overleftarrow{\nabla}_1 = \frac{1-q^{-2H}}{1-q^{-2}}, & \overleftarrow{\nabla}_2 = q^{-H/2}T_2, \\ \overleftarrow{\nabla}_4 = N, & \overleftarrow{\nabla}_3 = q^{-H/2}T_3 \end{array} \tag{25}$$

we obtain the algebra $U_qGL(2,C)$ in the form of the Drinfeld–Jimbo algebra.

$$\begin{array}{cc} [T_3, T_2] = \frac{q^H - q^{-H}}{q - q^{-1}}, & [H, T_3] = 2T_3 \\ [H, T_2] = -2T_2 & [N, H] = [N, T_2] = [N, T_3] = 0 \end{array} \tag{26}$$

Using the commutation relations between the parameters and 1-forms we obtained the C.R. between the parameters and its left differentials.

$$\begin{array}{l} da\ a = q^{-2}\ a\ da + \frac{q^2-1}{2q^2}\ a^2\ \mathrm{Tr}_q\omega, \\ dc\ c = q^{-2}\ c\ dc + \frac{q^2-1}{2q^2}\ c^2\ \mathrm{Tr}_q\omega, \\ da\ c = q^{-1}\ c\ da + \frac{q^2-1}{2q^2}\ a\ c\ \mathrm{Tr}_q\omega, \\ dc\ a = q^{-1}\ a\ dc + (q^{-2}-1)\ c\ da + \frac{q^2-1}{2q^2}\ c\ a\ \mathrm{Tr}_q\omega, \\ db\ b = q^2\ b\ db + \frac{1-q^2}{2}\ b^2\ \mathrm{Tr}_q\omega, \\ d(d)\ d = q^2\ d\ (d) + \frac{1-q^2}{2}\ d^2\ \mathrm{Tr}_q\omega, \\ db\ d = q\ d\ db + (q^2-1)\ b\ d(d) + \frac{1-q^2}{2}\ b\ d\ \mathrm{Tr}_q\omega, \\ d(d)\ b = q\ b\ d(d) + \frac{1-q^2}{2}\ d\ b\ \mathrm{Tr}_q\omega, \end{array} \tag{27}$$

$$\begin{array}{l} da\ b = q\ b\ da + \frac{(q^2-1)}{q^2 D_q}\ a\ b\ (qc\ db - a\ d(d)) + \frac{q^2-1}{2q^2}\ a\ b\ \mathrm{Tr}_q\omega, \\ da\ d = d\ da + \lambda\ b\ dc + \frac{q^2-1}{D_q}\ a\ d(d\ da - \frac{1}{q}\ b\ dc) - \frac{q^2-1}{2}\ a\ d\ \mathrm{Tr}_q\omega \\ dc\ b = b\ dc + \frac{(q^2-1)}{D_q}\ c\ b\ (d\ da - \frac{1}{q}\ b\ dc) - \frac{(q^2-1)}{2}\ c\ b\ \mathrm{Tr}_q\omega, \\ dc\ d = q\ d\ dc + \frac{(q^2-1)}{D_q}\ c\ d\ (d\ da - \frac{1}{q}\ b\ dc) - \frac{(q^2-1)}{2}\ c\ d\ \mathrm{Tr}_q\omega, \\ db\ a = q^{-1}\ a\ db + \frac{(q^2-1)}{q^2 D_q}\ b\ a\ (q\ c\ db - a\ d(d)) + \frac{(q^2-1)}{2q^2}\ b\ a\ \mathrm{Tr}_q\omega, \\ db\ c = c\ db + \frac{(q^2-1)}{D_q}\ b\ c\ (d\ da - \frac{1}{q}\ b\ dc) - \frac{(q^2-1)}{2}\ b\ c\ \mathrm{Tr}_q\omega, \\ d(d)\ a = a\ d(d) - \lambda\ c\ db + \frac{q^2-1}{D_q}\ d\ a\ (d\ da - \frac{1}{q}\ b\ dc) - \frac{(q^2-1)}{2}\ d\ a\ \mathrm{Tr}_q\omega \\ d(d)\ c = q^{-1}\ c\ d(d) + \frac{(q^2-1)}{D_q}\ d\ c\ (d\ da - \frac{1}{q}\ b\ dc) - \frac{(q^2-1)}{2}\ d\ c\ \mathrm{Tr}_q\omega \end{array} \tag{28}$$

III. Differential calculus on the subgroups of the $GL_q(2,C)$ and on the their real forms.

To obtain the differential calculus on the Borel subgroups $B_L^{(2)}(C)$ and $B_U^{(2)}(C)$ of the lower and the upper triangular matrices it is necessary to use in the formulas (20,21,23,27,28) the surjections $\pi : GLq(2,C) \to B_L^{(2)}(C)$ such that $\pi(b) = 0, \pi(\omega^2) = 0, \pi(\overleftarrow{\nabla}_2) = 0$ and $\pi : GLq(2,C) \to B_U^{(2)}(C)$ such that $\pi(c) = 0, \pi(\omega^3) = 0, \pi(\overleftarrow{\nabla}_3) = 0$. For the unitary matrices g and antihermitian ω^k: $g^+ = g^{-1}, \omega^{k+} = -\omega^k$.

$$g = \begin{pmatrix} a & -qD_q c^* \\ c & D_q a^* \end{pmatrix}, \quad \begin{array}{cc} aa^* + q^2 cc^* = 1, & a^*a + c^*c = 1, \\ cc^* = c^*c & D_q D_q^* = 1, \end{array} \quad q^* = q \tag{29}$$

the formulas (20,21,23,27,28) define the left differential calculus on the $Uq(2)$ group.
To obtain the left differential calculus on the $SL_q(2,C)$ it is necessary to suppose additional constraints $D_q = 1$ and $dD_q = D_q \mathrm{Tr}_q \omega = 0$. It is possible, so the quantum determinant D_q commutes with the 1-forms and $dD_q = 0$ is satisfied by vanishing $\bar{\omega^1} = \mathrm{Tr}_q\omega = \alpha(q\omega^1 + \frac{1}{q}\omega^4) = 0$. As a consequence of these conditions we obtained three-dimensional differential calculus, which is independent of the parameters α and β. Thus, we have the next commutation relations on the $SL_q(2,C)$ group

$$\begin{array}{lll} \omega^1 a = q^{-2} a\omega^1 & \omega^1 d = q^2 d\omega^1 & (\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 = 0 \\ \omega^2 a = q^{-1} a\omega^2 & \omega^2 d = q d\omega^2 & \omega^1\omega^2 + q^4\omega^2\omega^1 = 0 \\ \omega^3 a = q^{-1} a\omega^3 & \omega^3 d = q d\omega^3 & \omega^1\omega^3 + q^{-4}\omega^3\omega^1 = 0 \\ & & \omega^2\omega^3 + q^{-2}\omega^3\omega^2 = 0 \end{array} \tag{30}$$

and other relations are made by interchanging $a \leftrightarrow c$, $d \leftrightarrow b$. The algebra of vector fields in this case has the following form

$$\begin{array}{lll} q^2\overleftarrow{\nabla}_1\overleftarrow{\nabla}_3 - q^{-2}\overleftarrow{\nabla}_3\overleftarrow{\nabla}_1 & = & (1+q^2)\overleftarrow{\nabla}_3 \\ q^2\overleftarrow{\nabla}_2\overleftarrow{\nabla}_1 - q^{-2}\overleftarrow{\nabla}_1\overleftarrow{\nabla}_2 & = & (1+q^2)\overleftarrow{\nabla}_2 \\ \overleftarrow{\nabla}_3\overleftarrow{\nabla}_2 - q^2\overleftarrow{\nabla}_2\overleftarrow{\nabla}_3 & = & \overleftarrow{\nabla}_1 \end{array} \tag{31}$$

The quantum group $Sp_q(2,C)$ is defined by condition $T^t J_2 = J_2 T^{-1}$, where

$$J_2 = \begin{pmatrix} 0 & q^{\frac{-1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} \tag{32}$$

and coincides with $SL_q(2,C)$ group.

It is possible to introduce the two solutions of the differential calculus of Wess - Zumino on the quantum plane $C_q(2|0)$ as the Hopf algebra surjections $\pi : SL_q(2,C) \to B_L(C) = SL_q(2,C) \cap B_L^{(2)}(C)$ such that $\pi(b) = 0$, $\pi(\omega^2) = 0$, $\pi(\overleftarrow{\nabla}_2) = 0$ and $\pi : SL_q(2,C) \to B_U(C) = SL_q(2,C) \cap B_U^{(2)}(C)$ such that $\pi(c) = 0$, $\pi(\omega^3) = 0$, $\pi(\overleftarrow{\nabla}_3) = 0$.

$$g_- = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix}, g_+ = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} x^{-1} & y \\ 0 & x \end{pmatrix} \tag{33}$$

The quantum group $SU_q(2) = SL_q(2,C) \cap U_q(2)$ and the differential calculus on this group can be obtain from the differential calculus of the $SL_q(2,C)$ group by the surjection $\pi : SL_q(2,C) \to SU_q(2)$ such that $\pi(d) = a^*$, $\pi(b) = -qc^*$.

There are subgroup $U_q(1) \otimes U_q(1)$: the diagonal subgroup of the $U_q(2)$ which obtained by the surjection $\pi : U_q(2) \to T_q(2,C)$ such that $\pi(c) = 0$.

$$T_q(2,C) = \begin{pmatrix} a & 0 \\ 0 & D_q a^* \end{pmatrix} \tag{34}$$

and the diagonal subgroup of $GL_q(2,C)$ ($\pi : GL_q(2,C) \to T_q(2,C)$ such that $\pi(b) = \pi(c) = 0$). The differential calculus on this groups is the deformed calculus.

The $SU_q(2)$ group has the $U_q(1)$ subgroups: the diagonal subgroup of $U_q(2)$

$$g = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} \tag{35}$$

and diagonal subgroup of $SL_q(2,C)$

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{36}$$

There are, also, two subgroups $Z_-^{(2)}(C)$ and $Z_+^{(2)}(C)$ which can be considered as cosets $Z_-^{(2)}(C) = GL_q(2,C)/B_U^{(2)}(C)$ and $Z_+^{(2)}(C) = GL_q(2,C)/B_L^{(2)}(C)$. The matrix elements a, b, d commute with c for the $Z_-^{(2)}(C)$ and matrix elements a, c, d with b for the $Z_+^{(2)}(C)$.

The quantum subgroup $Sp_q(2) = SU_q(2) \cap Sp_q(2,C)$ coincides with the $SU_q(2)$ group.

The quantum subgroups $O_q(2,C), SO_q(2,C)$ and $SO_q(1,1)$ are the discrete group Z_4, Z_2.

The real forms of the quantum group $GL_q(2,C)$ and of its subgroups and the differential calculus on this group and subgroups can be obtained taking into account the conditions $a^+ = a$, $b^+ = b$, $c^+ = c$, $d^+ = d$. Thus we can obtain the differential calculi on the quantum group $GL_q(2,R)$ and on its quantum subgroups $B_L^{(2)}(R)$, $B_U^{(2)}(R)$, $SL_q(2,R)$, $SP_q(2,R)$, $B_L(R)$, $B_U(R)$ and on the real quantum plane $R_q(2|0)$.

IV. WZNW model $SL_q(2,R)$ and σ-models on the quantum group manifolds $SL_q(2,R)/U_h(1), C_q(2|9)$ and infinitesimal transformations on the quantum group space.

`1. Differential calculus on the` $SL_q(2,R)$

The matrix quantum group [4] $G = SL_q(2,R)$ is defined by the q-commutation relations (C.R.) of its group parameters. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{matrix} ab = qba, & cd = qdc, & cb = bc \\ ac = qca, & cd = qdc, & ad = da + (q - q^{-1})bc \end{matrix} \tag{37}$$

a, b, c, d - hermitian, $|q| = 1, Det_q g = ad - qbc = 1$.
For any elements $g,\ g' \in SL_q(2,R)$ element $g'' = g'g$ will belong to $SLq(2,R)$ if $a^{k'}a^l = a^l a^{k'}$. In the Gauss decomposition [7]

$$g = \begin{pmatrix} 1 & \varphi_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_+ & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho + \varphi_- \varphi_+ \rho & \varphi_- \rho^{-1} \\ \varphi_+ \rho & \rho^{-1} \end{pmatrix} \tag{38}$$

the C.R. are:

$$\rho\varphi_\pm = q\varphi_\pm\rho, \quad \varphi_-\varphi_+ = q^2\varphi_+\varphi_- \tag{39}$$

Let the quantum group is a manifolds of any possible transformations $g' = gg_0$. There are two kinds of the variation: the variation in the neighborhood of the arbitrary point of the group space $g' = g + dg$ and variation in the neighborhood of the unit of the group $g = 1 + \delta g$. First variation defines the group invariants: element of the distance between two neighboring points, element of the volume around the point. Second variation defines the group symmetry of this invariants. The C.R. between the variation dg and g define the type of the differential calculus.

Let $\omega = g^{-1}dg$ is the left differential Maurer-Cartan 1-form

$$\omega = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix}, \quad \mathrm{Tr}_q\omega = q^2\omega^1 + \omega^4 = 0 \tag{40}$$

The second order differential calculus on the $SL_q(2,R)$ group is defined by the C.R.

$$\begin{array}{ccc} \omega^1\rho = \frac{1}{q^2}\rho\omega^1, & \omega^2\rho = \frac{1}{q}\rho\omega^2, & \omega^3\rho = \frac{1}{q}\rho\omega^3 \\ \omega^1\varphi_\pm = \varphi_\pm\omega^1, & \omega^2\varphi_\pm = \varphi_\pm\omega^2, & \omega^3\varphi_\pm = \varphi_\pm\omega^3 \end{array} \tag{41}$$

The C.R. between the group parameters and their differentials are more complicated:

$$\begin{array}{cc} d\rho\rho = \frac{1}{q^2}\rho d\rho, & d\varphi_+\varphi_+ = \frac{1}{q^2}\varphi_+ d\varphi_+ + (q^4-1)\varphi_+^3 d\varphi_- \\ d\varphi_-\varphi_- = q^2\varphi_- d\varphi_-, & d\rho\varphi_- = q\varphi_- d\rho \end{array} \tag{42}$$

$$\begin{array}{ccc} d\varphi_-\varphi_+ = q^2\varphi_+ d\varphi_-, & d\varphi_+\varphi_- = \frac{1}{q^2}\varphi_- d\varphi_+, & d\rho d\varphi_- = -q d\varphi_- d\rho \\ d\varphi_- d\varphi_+ = -q^2 d\varphi_+ d\varphi_-, & d\varphi_-\rho = \frac{1}{q}\rho d\varphi_- & \end{array} \tag{43}$$

$$\begin{array}{c} d\varphi_+\rho = \frac{1}{q}\rho d\varphi_+ - q(q^2-1)\varphi_+^2\rho d\varphi_-, \quad d\rho\varphi_+ = q\varphi_+ d\rho - q^2(q^2-1)\varphi_+^2\rho d\varphi_- \\ d\rho d\varphi_+ + q d\varphi_+ d\rho + q^3(q^2-1)\varphi_+^2 d\varphi_- d\rho - \frac{(q^4-1)}{q^3}\varphi_+\rho d\varphi_- d\varphi_+ = 0 \end{array} \tag{44}$$

The Maurer-Cartan 1-forms are:

$$\omega^1 = \rho^{-1}d\rho + \varphi_+ d\varphi_-, \;\; \omega^3 = \frac{1}{q}\rho^2 d\varphi_+ - q^5\varphi_+^2\rho^2 d\varphi_-, \;\; \omega^2 = q\rho^{-2}d\varphi_- \tag{45}$$

$$\begin{array}{cc} (\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 = 0 & \omega^4 = -q^2\omega^1 \\ \omega^1\omega^2 + q^4\omega^2\omega^1 = 0 & \omega^1\omega^3 + q^{-4}\omega^3\omega^1 = 0 \\ \omega^2\omega^3 + q^{-2}\omega^3\omega^2 = 0 & \end{array} \tag{46}$$

The left vector fields $\overleftarrow{\nabla}_k$ can be obtained from the applying the left differential to an arbitrary function on the quantum group $df = (f\frac{\overleftarrow{\partial}}{\partial a^k})da^k = (f\overleftarrow{\nabla}_k)\omega^k$.

$$\overleftarrow{\nabla} = \begin{pmatrix} \overleftarrow{\nabla}_1 & \overleftarrow{\nabla}_2 \\ \overleftarrow{\nabla}_3 & \overleftarrow{\nabla}_4 \end{pmatrix} \qquad \hat{\overleftarrow{\nabla}}_1 = \overleftarrow{\nabla}_1 - q^2\overleftarrow{\nabla}_4 \qquad \hat{\overleftarrow{\nabla}}_4 = \overleftarrow{\nabla}_1 + \overleftarrow{\nabla}_4 \tag{47}$$

The C.R. for vector fields have the following form

$$\begin{array}{ccc} \rho\hat{\overleftarrow{\nabla}}_1 = \frac{1}{q^2}\hat{\overleftarrow{\nabla}}_1\rho + \rho, & \varphi_-\hat{\overleftarrow{\nabla}}_1 = \hat{\overleftarrow{\nabla}}_1\varphi_-, & \varphi_+\hat{\overleftarrow{\nabla}}_1 = \hat{\overleftarrow{\nabla}}_1\varphi_+, \\ \rho\overleftarrow{\nabla}_2 = \frac{1}{q}\overleftarrow{\nabla}_2\rho - \varphi_+\rho^3, & \varphi_-\overleftarrow{\nabla}_2 = \overleftarrow{\nabla}_2\varphi_- + \frac{1}{q}\rho^2, & \varphi_+\overleftarrow{\nabla}_2 = \overleftarrow{\nabla}_2\varphi_+ + q\varphi_+^2\rho^2 \\ \rho\overleftarrow{\nabla}_3 = \frac{1}{q}\overleftarrow{\nabla}_3\rho, & \varphi_-\overleftarrow{\nabla}_3 = \overleftarrow{\nabla}_3\varphi_-, & \varphi_+\overleftarrow{\nabla}_3 = \overleftarrow{\nabla}_3\varphi_+ + q\rho^{-2}, \end{array} \tag{48}$$

$$\begin{array}{c} q^2\hat{\overleftarrow{\nabla}}_1\overleftarrow{\nabla}_3 - \frac{1}{q^2}\overleftarrow{\nabla}_3\hat{\overleftarrow{\nabla}}_1 = (q^2+1)\overleftarrow{\nabla}_3, \\ q^2\overleftarrow{\nabla}_2\hat{\overleftarrow{\nabla}}_1 - \frac{1}{q^2}\hat{\overleftarrow{\nabla}}_1\overleftarrow{\nabla}_2 = (q^2+1)\overleftarrow{\nabla}_2, \\ \overleftarrow{\nabla}_3\overleftarrow{\nabla}_2 - \frac{1}{q^2}\overleftarrow{\nabla}_2\overleftarrow{\nabla}_3 = \hat{\overleftarrow{\nabla}}_1 \end{array} \tag{49}$$

and $\overleftarrow{\nabla}_k$ have the form

$$\hat{\overleftarrow{\nabla}}_1 = \frac{\overleftarrow{\partial}}{\partial\rho}\rho, \;\; \overleftarrow{\nabla}_2 = \frac{1}{q}\frac{\overleftarrow{\partial}}{\partial\varphi_-}\rho^2 - \frac{\overleftarrow{\partial}}{\partial\rho}\varphi_+\rho^3 + q\frac{\overleftarrow{\partial}}{\partial\varphi_+}\varphi_+^2\rho^2, \;\; \overleftarrow{\nabla}_3 = q\frac{\overleftarrow{\partial}}{\partial\varphi_+}\rho^{-2} \tag{50}$$

The left vector fields and the left derivatives act on the any function of the group parameters from the right side.

2.WZNW model on the $SL_q(2,R)$ group.

The existing of the quantum group structure in the WZNW model was shown in [8,9]. The σ-models on the quantum group spaces was considered in [7,10-12]. To construct the WZNW model with $SL_q(2,R)$ group symmetry, we consider the space $M^{(1,1)} \oplus SL_q(2,R)$, where base space $M^{(1,1)}$ is the commutative (nondeformed) space. The element of the volume in $M^{(1,1)}$ space, which is the invariant of $SL_q(2,R)$ group, is

$$\frac{\mathrm{Tr}_q[\omega(d)\wedge dz^\mu][\omega(d)\wedge dz^\mu]}{2\epsilon^{\lambda\rho}dz^\lambda \wedge dz^\rho} = \mathrm{Tr}_q(\omega_\mu\omega^\mu)d^2z, \tag{51}$$

where $\omega(d) = \omega_\mu dz^\mu, z^\mu \epsilon M^{1,1}, \mu = 1,2$. For any 2×2 matrix $A, \mathrm{Tr}_q A = q^2A^1 + A^4$. As a result we have

$$\mathrm{Tr}_q(\omega_\mu\omega^\mu) = q^5[2]_q\rho^{-2}\partial_\mu\rho\partial^\mu\rho + q^5[2]_q\rho^{-1}\varphi_+(\partial_\mu\varphi_-\partial^\mu\rho + \frac{1}{q}\partial_\mu\rho\partial^\mu\varphi_-)+$$

$$(\partial_\mu\varphi_-\partial^\mu\varphi_+ + q^2\partial_\mu\varphi_+\partial^\mu\varphi_-) - q^2(q^4-1)\varphi_+^2\partial_\mu\varphi_-\partial^\mu\varphi_-$$

The C.R. are now in the same space-time point and $d\rho = \partial_\mu\rho dz^\mu, d\varphi_\pm = \partial_\mu\varphi_\pm dz^\mu$ and $[n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}$. The Wess-Zumino term

$$Tr_q(\omega(d)\wedge\omega(d)\wedge\omega(d)) = \frac{q[2]_q[3]_q}{6}\varepsilon^{\mu\nu\lambda}\partial_\lambda(\rho^{-1}\partial_\mu\rho\partial_\nu\varphi_-\varphi_+)d^3z \tag{52}$$

is the total derivative. Finally, the WZNW-action with the $SL_q(2,R)$ quantum group symmetry describes the 2-dimensional relativistic string in the background gravity and antisymmetric fields

$$S[\rho,\varphi_-,\varphi_+] = \frac{k}{4\pi}\int d^2z(G_{AB}\partial_\mu X^A\partial^\mu X^B + B_{AB}\varepsilon_{\mu\nu}\partial^\mu X^A\partial^\nu X^B), \tag{53}$$

where $X^A = (\rho,\varphi_-,\varphi_+)$ and the background gravity and antisymmetric fields have the following form:

$$G_{AB} = \begin{pmatrix} q^5[2]_q\rho^{-2} & q^4[2]_q\rho^{-1}\varphi_+ & 0 \\ q^5[2]_q\rho^{-1}\varphi_+ & -q^2(q^4-1)\varphi_+^2 & 1 \\ 0 & q^2 & 0 \end{pmatrix} \tag{54}$$

$$B_{AB} = \frac{q^3[2]_q[3]_q}{6}\varphi_+\rho^{-1}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{55}$$

The group symmetry of this model is $SL_q(2,R)\otimes SL_q(2,R)$, because under the left multiplication on the group $g' = g_0 g$ the differential forms of Cartan are invariant, $\omega' = \omega$, and under the right multiplication $g' = gg_0$ the differential forms are covariant, $\omega' = g_0^{-1}\omega g_0$. But $Tr_q A$ is invariant of the transformation $A' = g_0^{-1}Ag_0$, because the elements of matrix A commute with the elements of matrix g_0, by definition of the quantum group. Therefore, this model describes the spontaneous breaking of the $SL_q(2,R)\otimes SL_q(2,R)$ symmetry to the $SL_q(2,R)$ one.

3.σ-model on the $SL_q(2,R)/U_h(1)$ group.

Let us consider the spontaneous breaking symmetry in the $\sigma-$ model with the $SL_q(2,R)/U_h(1)$ group symmetry. Let $G = KH$, K-coset, H-subgroup. The Maurer-Cartan 1-forms

$$k^{-1}dk = \begin{pmatrix} q^2\varphi_+ d\varphi_- & d\varphi- \\ d\varphi_+ - q^2\varphi_+^2 d\varphi_- & -\varphi_+ d\varphi_- \end{pmatrix} = \omega + \theta, \tag{56}$$

where $\omega\epsilon K$, $\theta\epsilon H$ and the coset elements $\varphi_\pm$ commute with the subgroup parameter ρ and satisfy to C.R. of $SL_q(2,R)$ group among themselves. There is a question: how does coset and subgroup separate from $k^{-1}dk$? In opposite to the classical case, there is the 3-parametric family of the $U(1)$ subgroups. The Lagrangian has the following form:

$$L_n = \frac{1}{2}Tr_q(\omega_\mu\omega^\mu) = \frac{(q^4+1)}{4q^4}(\partial_\mu\varphi_-\partial^\mu\varphi_+ + q^2\partial_\mu\varphi_+\partial^\mu\varphi_-) - c_n(q)\varphi_+^2\partial_\mu\varphi_-\partial^\mu\varphi_-, \tag{57}$$

where $c_n(q)$ depends on the choice of a subgroup. There are three most interesting examples.

1). Nondeformed $U(1)$ subgroup : $c_1 = \frac{2q^4-q^2+1}{2}$

$$\omega = \begin{pmatrix} (q^2-1)\varphi_+ d\varphi_- & d\varphi_- \\ d\varphi_+ - q^2\varphi_+^2 d\varphi_- & 0 \end{pmatrix}, \quad \theta = \varphi_+ d\varphi_- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{58}$$

The algebra symmetry of this Lagrangian is defined by the Maurer-Cartan equations:

$$d\theta = -\begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix}\omega\omega + (q^2-1)\omega\theta, d\omega = -\begin{pmatrix} \frac{q^2-1}{q^2} & 0 \\ 0 & 0 \end{pmatrix}\omega\omega - q^3[2]_q\omega\theta, \theta\omega = q^4\omega\theta \tag{59}$$

The C.R. between the coset and the subgroup forms are common for all of the examples

$$\begin{aligned} \omega^1\omega^3 + q^4\omega^3\omega^1 = 0, \quad \omega^2\omega^3 + q^2\omega^3\omega^2 = 0 \\ \omega^1\omega^3 + q^4\omega^3\omega^1 = 0, \quad \omega^4\omega^3 + q^4\omega^3\omega^4 = 0 \end{aligned} \tag{60}$$

2). Classical coset structure: $c_2 = \frac{q^6+1}{4}$

$$\omega = \begin{pmatrix} 0 & d\varphi_- \\ d\varphi_+ - q^2\varphi_+^2 d\varphi_- & 0 \end{pmatrix}, \quad \theta = \varphi_+ d\varphi_- \begin{pmatrix} q^2 & 0 \\ 0 & -1 \end{pmatrix} \tag{61}$$

$$d\theta = -\omega\omega, \ d\omega = -\omega\theta - \theta\omega, \ \theta\omega = q^2\omega\theta \tag{62}$$

3). There is one of the examples of the 2- parametric family $U_q(1)$ subgroups:

$$c_3 = \frac{2q^4-q^2+1}{2q^2}, \quad \omega = \begin{pmatrix} \frac{(q^2-1)}{q^2}\varphi_+ d\varphi_- & d\varphi_- \\ d\varphi_+ - q^2\varphi_+^2 d\varphi_- & 0 \end{pmatrix}, \quad \theta = \frac{1}{q^2}\varphi_+ d\varphi_- \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix} \tag{63}$$

$$d\theta = -\begin{pmatrix} q^{-4} & 0 \\ 0 & 1 \end{pmatrix}\omega\omega + (q^4-1)\omega\theta, \ d\omega = -\begin{pmatrix} \frac{q^4-1}{q^4} & 0 \\ 0 & 0 \end{pmatrix}\omega\omega - q^4[2]_q\omega\theta, \theta\omega = q^6\omega\theta \tag{64}$$

Why we have obtained different algebras of a symmetry for the same subgroup? That is possible because we can use the different map from the algebra to the group, for example:

$$g = \exp(\varphi_-\tau_+)\exp(\varphi_+\tau_-)\exp(\ln\rho\tau_3), \tag{65}$$

where τ are the Pauli matrices – the fundamental representation of the $U_q(SL(2,R)$ algebra.The group stability of the vacuum is $U(1)$. In the another parametrization

$$g = \exp(\varphi_-\tau_+)\exp(\varphi_+\tau_-)(1-\frac{(q^2-1)}{q^2}\nabla_3)^{\frac{\ln\rho}{\ln q^{-2}}},\ \nabla_3 = \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix} \tag{66}$$

the group stability of the vacuum is $U_q(1)$. The group symmetry of this Lagrangians is $SL_q(2,R)$ spontaneously broken to $U_p(1)$, $\quad p = q^{\pm 2n}, n = 0, 1....$ Under the left multiplication on the group $g' = g_0 g$, the differential form $g^{-1}dg = h^{-1}(\omega+\theta)h = g'^{-1}dg'$. Therefore, $\omega'+\theta' = h'h^{-1}(\omega+\theta)hh'^{-1}$. Again, the decomposition on the coset and the subgroup forms is not unique after the transformation . The group transformation can transform the Lagrangian with the $U_{p_1}(1)$ subgroup of the vacuum stability to the Lagrangian with the $U_{p_2}(1)$ subgroup.

`4.Variational calculus on the` $SL_q(2,R)$ `group.`

It is possible to obtain the variational calculus on the group by two ways: from the C.R. between the left vector fields and group parameters and from the infinitesimal transformations on the group. Let us multiply the C.R. (48) between ∇_n and group parameters on the parameters of transformation R^n. The form of the infinitesimal transformations of the group parameters is obtained under the requirement

$$[X_A, \overleftarrow{\nabla}_n R^n] = X_A\overleftarrow{\delta} = X_A\sum_{n=1}^{3}\overleftarrow{\delta}_{R_n}, \quad X_A = (\rho, \varphi_-, \varphi_+), \quad [A,B] = AB - BA \tag{67}$$

By imposing the C.R. between the parameters of infinitesimal transformations and group parameters

$$\begin{array}{lll} \rho R^1 = q^2 R^1\rho, & \varphi_- R^1 = R^1\varphi_-, & \varphi_+ R^1 = R^1\varphi_+ \\ \rho R^2 = q R^2\rho, & \varphi_- R^2 = R^2\varphi_-, & \varphi_+ R^2 = R^2\varphi_+ \\ \rho R^3 = q R^3\rho, & \varphi_- R^3 = R^3\varphi_-, & \varphi_+ R^3 = R^3\varphi_+ \end{array} \tag{68}$$

we obtain the infinitesimal transformation of the group parameters

$$\rho\overleftarrow{\delta} = \rho R^1 - \varphi_+\rho^3 R^2, \quad \varphi_-\overleftarrow{\delta} = \tfrac{1}{q}\rho^2 R^2, \quad \varphi_+\overleftarrow{\delta} = q\varphi_+^2\rho^2 R^2 + q\rho^{-2}R^3 \tag{69}$$

In the terms of the components $\overleftarrow{\delta}_{R_n}$ the infinitesimal transformations have the following form:

$$\begin{array}{lll} \rho\overleftarrow{\delta}_{R^1} = \rho R^1, & \rho\overleftarrow{\delta}_{R^2} = -\varphi_+\rho^3 R^2, & \rho\overleftarrow{\delta}_{R^3} = 0 \\ \varphi_-\overleftarrow{\delta}_{R^1} = 0, & \varphi_-\overleftarrow{\delta}_{R^2} = \frac{1}{q}\rho^2 R^2, & \varphi_-\overleftarrow{\delta}_{R^3} = 0 \\ \varphi_+\overleftarrow{\delta}_{R^1} = 0, & \varphi_+\overleftarrow{\delta}_{R^2} = q{\varphi_+}^2\rho^2 R^2, & \varphi_+\overleftarrow{\delta}_{R^3} = q\rho^{-2}R^3 \end{array} \tag{70}$$

We postulate the $U_qSL(2,R)$ algebra of the vector fields in the form, which it is common for a boson theory and a supersymmetric theory

$$\begin{array}{c} [\overleftarrow{\nabla}_1 R^1, \overleftarrow{\nabla}_2 R^2] = (q^2+1)\overleftarrow{\nabla}_2 R^1 R^2, \\ [\overleftarrow{\nabla}_3 R^3, \overleftarrow{\nabla}_1 R^1] = (q^2+1)\overleftarrow{\nabla}_3 R^3 R^1, \\ [\overleftarrow{\nabla}_2 R^2, \overleftarrow{\nabla}_3 R^3] = \overleftarrow{\nabla}_1 R^2 R^3 \end{array} \tag{71}$$

It is possible, if we suppose the following C.R. between the infinitesimal parameters and the vector fields:

$$\begin{array}{ccc} R^1\overleftarrow{\nabla}_2 = \frac{1}{q^4}\overleftarrow{\nabla}_2 R^1 & R^2\overleftarrow{\nabla}_1 = q^4\overleftarrow{\nabla}_1 R^2 + q^2(q^4-1)R^2 & R^1\overleftarrow{\nabla}_3 = q^4\overleftarrow{\nabla}_3 R^1 \\ R^3\overleftarrow{\nabla}_1 = \frac{1}{q^4}\overleftarrow{\nabla}_1 R^3 + \frac{q^4-1}{q^4}R^3 & R^3\overleftarrow{\nabla}_2 = \overleftarrow{\nabla}_2 R^3 & R^2\overleftarrow{\nabla}_3 = \overleftarrow{\nabla}_3 R^2 \end{array} \tag{72}$$

It means, that we have the same C.R. between basis of 1-forms and basis of vector fields

$$\begin{array}{ccc} \omega^1\overleftarrow{\nabla}_2 = \frac{1}{q^4}\overleftarrow{\nabla}_2\omega^1 & \omega^2\overleftarrow{\nabla}_1 = q^4\overleftarrow{\nabla}_1\omega^2 + q^2(q^4-1)\omega^2 & \omega^1\overleftarrow{\nabla}_3 = q^4\overleftarrow{\nabla}_3\omega^1 \\ \omega^3\overleftarrow{\nabla}_1 = \frac{1}{q^4}\overleftarrow{\nabla}_1\omega^3 + \frac{q^4-1}{q^4}\omega^3 & \omega^3\overleftarrow{\nabla}_2 = \overleftarrow{\nabla}_2\omega^3 & R^2\overleftarrow{\nabla}_3 = \overleftarrow{\nabla}_3\omega^2 \end{array} \tag{73}$$

The same result we can obtain from the right infinitesimal multiplication on the group $g' = gg_0$, where $g_0 = 1 + \delta g_0$. For

$$\delta g_0 = \begin{pmatrix} R^1 & R^2 \\ R^3 & -q^2R^1 \end{pmatrix} \tag{74}$$

we see, that $dg = g\delta g_0$ and C.R. for R^n have the following form:

$$R^1R^2 = q^4R^2R^1, \quad R^3R^1 = q^4R^1R^3, \quad R^3R^2 = q^2R^2R^3 \tag{75}$$

simultaneously with the condition $R^4 = -q^2R^1$. The C.R. of the variational calculus

$$\begin{array}{cc} (\rho\overleftarrow{\delta})\rho = \frac{1}{q^2}\rho(\rho\overleftarrow{\delta}), & (\varphi_+\overleftarrow{\delta})\varphi_+ = \frac{1}{q^2}\varphi_+(\varphi_+\overleftarrow{\delta}) + (q^4-1)\varphi_+^3(\varphi_-\overleftarrow{\delta}) \\ (\varphi_-\overleftarrow{\delta})\varphi_- = q^2\varphi_-(\varphi_-\overleftarrow{\delta}), & (\rho\overleftarrow{\delta})\varphi_- = q\varphi_-(\rho\overleftarrow{\delta}) \end{array} \tag{76}$$

$$(\varphi_-\overleftarrow{\delta})\rho = \frac{1}{q}\rho(\varphi_-\overleftarrow{\delta}) \tag{77}$$

$$\begin{array}{cc} (\varphi_+\overleftarrow{\delta})\rho = \frac{1}{q}\rho(\varphi_+\overleftarrow{\delta}) - q(q^2-1)\varphi_+^2\rho(\varphi_-\overleftarrow{\delta}), & (\varphi_-\overleftarrow{\delta})\varphi_+ = q^2\varphi_+(\varphi_-\overleftarrow{\delta}) \\ (\rho\overleftarrow{\delta})\varphi_+ = q\varphi_+(\rho\overleftarrow{\delta}) - q^2(q^2-1)\varphi_+^2\rho(\varphi_-\overleftarrow{\delta}), & (\varphi_+\overleftarrow{\delta})\varphi_- = \frac{1}{q^2}\varphi_-(\varphi_+\overleftarrow{\delta}) \end{array} \tag{78}$$

are consistent with the C.R. (39) and are coincide with the C.R. of the differential calculus (42,43). The $U_q(SL(2,R))$ algebra in the form (71) is the condition of the compatibility of the relations (69,70)

$$\begin{array}{c} X^A[\overleftarrow{\delta}_{R^1}, \overleftarrow{\delta}_{R^2}] = X^A(q^2+1)\overleftarrow{\delta}_{R^1R^2} \\ X^A[\overleftarrow{\delta}_{R^3}, \overleftarrow{\delta}_{R^1}] = X^A(q^2+1)\overleftarrow{\delta}_{R^3R^1} \\ X^A[\overleftarrow{\delta}_{R^2}, \overleftarrow{\delta}_{R^3}] = X^A\overleftarrow{\delta}_{R^2R^3} \end{array} \tag{79}$$

5.Equations of motion

We use the extremum principle of the action to obtain the equations of motion and we must to commute the variations of fields and their derivatives on the right or on the left side. We can use both variation dX^A and $X^A\overleftarrow{\delta}$ to do this. The C.R. of the differential calculus on the $SL_q(2,R)$ group are insufficient to do ordering. Therefore, we need in the differential calculus on the Lagrangian manifolds $(\rho, \varphi_\pm, \dot{\rho}, \dot{\varphi}_\pm, \acute{\rho}, \acute{\varphi}_\pm)$. This is not the quantum group manifold and we can not use the formalism of 1-forms. We can require, that the Lagrangian equation of motion be coincident with the conservation law $\partial_\mu\omega^\mu = 0$ for Lagrangian with $SL_q(2,R)$ group symmetry. At last, we can investigate the 1- dimensional σ- models. The variational calculus is more suitable to obtain the equations of motion. The C.R. between the X^A, $\dot{X}^A$, $\acute{X}^A$ and R^n, $\dot{R}^n$, $\acute{R}^n$ can obtain by differentiating the relations (68).

$$\begin{array}{ccc} \dot{\rho}R^1 = q^2R^1\dot{\rho} & \dot{\rho}\dot{R}^1 = q^2\dot{R}^1\dot{\rho} & \rho\dot{R}^1 = \dot{R}^1\rho \\ \dot{\rho}R^2 = qR^2\dot{\rho} & \dot{\rho}\dot{R}^2 = q\dot{R}^2\dot{\rho} & \rho\dot{R}^2 = q\dot{R}^2\rho \\ \dot{\rho}R^3 = qR^3\dot{\rho} & \dot{\rho}\dot{R}^3 = q\dot{R}^3\dot{\rho} & \rho\dot{R}^3 = q\dot{R}^3\rho \end{array} \tag{80}$$

The derivatives of $\varphi_\pm$ commute with the derivatives of R^n.

`6. One dimensional` σ`- model on the quantum plane` $C_q(2|0)$`.`

The differential calculus on the $C_q(2|0)$ is coincide with the differential calculus on the Borel subgroup of $SL_q(2,C)$ and can be obtained from the differential calculus on the $SL_q(2,C)$ by surjection: π:$SL_q(2,C)\to B_L$ such that $\pi(b)=0$.

$$g=\begin{pmatrix} x & 0 \\ y & x^{-1}\end{pmatrix},\quad \begin{array}{cc} xy=qyx & \dot{y}y=q^{-2}y\dot{y} \\ \dot{x}x=q^{-2}x\dot{x} & \dot{x}y=q^{-1}y\dot{x}\end{array} \tag{81}$$

$$\omega=\begin{pmatrix} x^{-1}dx & 0 \\ xdy-qydx & -q^2x^{-1}dx\end{pmatrix},\ \dot{y}x=q^{-1}x\dot{y}-\frac{(q^2-1)}{q^2}y\dot{x} \tag{82}$$

In term of the variables $\rho,\varphi_\pm$

$$g=\begin{pmatrix} \rho & 0 \\ \varphi_+\rho & \rho^{-1}\end{pmatrix};\quad \omega=\begin{pmatrix} \rho^{-1}d\rho & 0 \\ \frac{1}{q}\rho^2 d\varphi_+ & -q^2\rho^{-1}d\rho\end{pmatrix} \tag{83}$$

$$L=\frac{1}{2}\mathrm{Tr}_q(\omega_\mu\omega^\mu)=\frac{q^4(q^2+1)}{2}\rho^{-2}\dot{\rho}^2 \tag{84}$$

The equation of motion $\dot{\omega}^1=\rho^{-1}\ddot{\rho}-q^2\rho^{-1}\dot{\rho}^2=0$ will coincide with Lagrangian equation , if we impose the C.R. $d\dot{\rho}\dot{\rho}=\frac{1}{q^2}\dot{\rho}d\dot{\rho}$. The classical solution of this equation is

$$\rho=\alpha\exp(\beta t),\quad \alpha\beta=q^2\beta\alpha \tag{85}$$

and C.R. in the different time points are

$$\rho(t)\rho(t')=\rho(q^2t')\rho(\frac{1}{q^2}t),\ \ \rho(t)\rho(t')=\exp[q^2(q^2-1)\beta[(t-t')]\rho(t')\rho(t) \tag{86}$$

There are 4×4 matrix representations of α,β such,that $det_q\alpha=0$ or $det_q\beta=0$.Therefore, we can rewrite this Lagrangian as a 4×4 matrix model for the commuting fields. In conclusion, we note that 2-dimensional σ-model on the guantum plane

$$L=\frac{q^4(q^2+1)}{2}\rho^{-2}\partial_\mu\rho\partial^\mu\rho \tag{87}$$

leads to the C.R. $d\acute{\rho}\acute{\rho}=\frac{1}{q^2}\acute{\rho}d\acute{\rho}$ and the equation of motion $\partial_\mu\partial^\mu\rho-q^2\rho^{-1}\partial_\mu\rho\partial^\mu\rho=0,\ \ \mu=1,2$.

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References

1. V. P. Akulov, V. D. Gershun, A. I. Gumenchuk, *JETP Lett.*, **58** (1993), 474
2. V. Akulov, V. Gershun, *q-alg 9509030*

3. V. P. Akulov, V. D. Gershun,*Proc. Int. conf. on math, phys. (Rahov, Ukraine, 1995)*, (in press)

4. L. Faddeev, N. Reshetikhin, L. Takhtajan, *Algebra i Analiz* **1** (1989), 178

5. A.Schirrmacher, J.Wess, B.Zumino, *Z.Phys.* **C49** (1990), 317

6. S. L. Woronowicz, *Comm. Math. Phys.* **122** (1989), 125

7. V. P. Akulov, V. D. Gershun, A. I. Gumenchuk, *JETP Lett.*, **56** (1992), 177

8. L. D. Fadeev, *Commun. Math, Phys.*, **132** (1990), 131

9. A. Alekseev and S. Shatashvili, *Commun. Math. Phys.*, **133** (1990), 353

10. I. Y. Aref'eva and I. V. Volovich, *Phys. Lett.*, **264B** (1991), 62

11. Y. Frishman, J. Lukierski, W. J. Zakrzewski, *J.Phys.A:Math.Gen*, **26** (1993), 301

12. V. D. Gershun, *Proc. Xth Int. conf. "Problems of Quantum Field Theory" (Alushta, Ukraine,1996)*, JINR, Dubna, (1996), 119

13. V. D. Gershun *q-alg 9709022*